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Newark, Delaware

Iterative Methods for Solving the Exterior Dirichlet Problem for the Helmholtz Equation with Applications to the Inverse Scattering Problem for Low Frequency Acoustic Waves

by

David Colton
Department of Mathematical Sciences
University of Delaware
Newark, Delaware 19711

and

Rainer Kress
Lehrstuhle fur Numerische und
Angewandte Mathematik
Universität Göttingen
Göttingen, West Germany

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I. Introduction

In a previous paper ([5]) one of us presented an iterative method for solving the exterior Dirichlet problem for the Helmholtz equation defined in the plane and used this result to provide a constructive approach for solving the low frequency inverse scattering problem for a cylinder. These results were based on the use of conformal mapping and the fact that the integral of the normal derivative of the total field over the boundary of the obstacle vanishes, neither of which is valid in the three dimensional case. ~~In~~ this paper, we shall show how the analysis of [5] can be modified in order to extend these results to the case of the exterior Dirichlet problem for the Helmholtz equation in $\mathbb{R}^3$. Our results are based on chosing an appropriate fundamental solution such that the integral equation associated with the exterior Dirichlet problem can be solved by iteration for sufficiently small values of the wave number. Previous results in this direction have been given by Kleinman ([9]) and Ahner ([1]). However, in Kleinman's approach it was necessary to first construct the Green's function for Laplace's equation defined in the exterior of the scattering obstacle D, whereas in Ahner's approach it was necessary to compute the first eigenfunction of the integral equation associated with this problem. Our method avoids both of these computations and instead provides an integral equation formulation of the exterior Dirichlet problem for the Helmholtz equation with the kernel of the integral equation being independent of D and expressable in closed form. Such a formulation is particularly suitable for obtaining analytic

approximation for low frequency scattering problems (c.f. [2]). However, rather than presenting an application to the direct scattering problem, we shall show how our iterative procedure can be used to obtain information on the inverse scattering problem for low frequency acoustic waves. The importance of our iterative procedure in this case is that it allows us to rigorously establish the limiting behaviour of the total field as the wave number k tends to zero. This step is in general nontrivial since in the classical formulation of the scattering problem as an integral equation the resolvant operator has a pole at $k = 0$ and we are thus faced with a singular perturbation problem (c.f. [12] and the references contained therein. In this regard we note that in many cases it is not in fact the limiting behaviour of the total field that is of interest, but instead the coefficients of certain higher order terms - c.f. [4], [5]). Having established the limiting static problem associated with the inverse scattering problem, we proceed to follow a standard approach (c.f. [16]) in order to obtain information on the shape of the scattering obstacle from a knowledge of the low frequency behaviour of the far field pattern. Our contribution here is to note that although in general the far field pattern is only known approximately, the determination of the shape of the obstacle D is stabilized by the fact that the solution of the static potential problem is known to have values lying between zero and one, i.e. the potential function to be determined lies in a compact set of harmonic functions (c.f. [8], [15]). Here it is necessary to assume that we know "a priori" the radius of a ball containing D in

its interior, with optimal results holding if we know the radius of the smallest such ball containing D.

II. The Exterior Dirichlet Problem for the Helmholtz Equation

We begin by first considering the exterior Dirichlet problem for Laplace's equation, i.e. to find $u \in C^2(\mathbb{R}^3\setminus\bar{D}) \cap C^0(\mathbb{R}^3\setminus D)$ such that

$$u(\underset{\sim}{x}) = u^i(\underset{\sim}{x}) + u^s(\underset{\sim}{x}) \quad \text{in} \quad \mathbb{R}^3\setminus D \tag{2.1a}$$

$$\Delta_3 u = 0 \quad \text{in} \quad \mathbb{R}^3\setminus\bar{D} \tag{2.1b}$$

$$u(\underset{\sim}{x}) = 0 \quad \text{for} \quad \underset{\sim}{x} \in \partial D \tag{2.1c}$$

$$\lim_{r\to\infty} u^s(\underset{\sim}{x}) = 0 \tag{2.1d}$$

where D is a bounded simply connected domain in $\mathbb{R}^3$ containing the origin with C^2 boundary ∂D, u^i is a given solution of Laplace's equation defined in all of $\mathbb{R}^3$, and $r = |\underset{\sim}{x}|$. We note that under these conditions a solution u to (2.1) exists and $u \in C^2(\mathbb{R}^3\setminus\bar{D}) \cap C^1(\mathbb{R}^3\setminus D)$ (c.f. [7]). Now let B be a ball of radius R contained in D with center at the origin and let

$$G(\underset{\sim}{x},\underset{\sim}{y}) = \frac{1}{|\underset{\sim}{x}-\underset{\sim}{y}|} - \frac{R}{|\underset{\sim}{x}|}\frac{1}{|\underset{\sim}{x}^*-\underset{\sim}{y}|} \; ; \; x^* = \frac{R^2}{|\underset{\sim}{x}|^2}\underset{\sim}{x} \tag{2.2}$$

be the Green's function for Laplace's equation in the exterior of B. Then from Green's formula we have

$$\frac{1}{4\pi}\int_{\partial D}\left\{G(\underset{\sim}{x},\underset{\sim}{y})\frac{\partial u^s}{\partial\nu(y)} - u^s\frac{\partial}{\partial\nu(y)}G(\underset{\sim}{x},\underset{\sim}{y})\right\}ds(y)$$

$$= \begin{cases} -1/2\, u^s(\underset{\sim}{x}), & x \in \partial D \\ -u^s(\underset{\sim}{x}), & \underset{\sim}{x} \in \mathbb{R}^3\setminus\bar{D} \end{cases} \tag{2.3}$$

where ν denotes the unit outward normal to ∂D. In addition, since u^i is a solution of Laplace's equation in all of $\mathbb{R}^3$ and $G(\underset{\sim}{x},\underset{\sim}{y}) = 0$ for $\underset{\sim}{y} \in \partial B$, we have

$$\frac{1}{4\pi}\int_{\partial D} \left\{G(\underset{\sim}{x},\underset{\sim}{y}) \frac{\partial u^i}{\partial \nu(y)} - u^i \frac{\partial}{\partial \nu(y)} G(\underset{\sim}{x},\underset{\sim}{y})\right\} ds(y)$$

$$+ \frac{1}{4\pi}\int_{\partial B} u^i \frac{\partial}{\partial \nu(y)} G(\underset{\sim}{x},\underset{\sim}{y}) \, ds(y) \tag{2.4}$$

$$= \begin{cases} 1/2 \; u^i(\underset{\sim}{x}), & \underset{\sim}{x} \in \partial D \\ 0 \;, & \underset{\sim}{x} \in \mathbb{R}^3 \backslash \bar{D} \end{cases}$$

where again ν denotes the unit outward normal. Adding (2.3) and (2.4) together and using the fact that $u(\underset{\sim}{x}) = 0$ for $\underset{\sim}{x} \in \partial D$ now gives

$$-u^i(\underset{\sim}{x}) + \frac{1}{4\pi}\int_{\partial D} G(\underset{\sim}{x},\underset{\sim}{y}) \frac{\partial u}{\partial \nu(y)} ds(y) + \frac{1}{4\pi}\int_{\partial B} u^i \frac{\partial}{\partial \nu(y)} G(\underset{\sim}{x},\underset{\sim}{y}) \, ds(y)$$

$$= \begin{cases} -u(\underset{\sim}{x}) \;, & \underset{\sim}{x} \in \mathbb{R}^3 \backslash \bar{D} \\ -1/2 u(\underset{\sim}{x}), & \underset{\sim}{x} \in \partial D \end{cases} \;. \tag{2.5}$$

From the jump discontinuity properties of the single layer potential we now have the following integral equation for $\partial u/\partial \nu$ evaluated on ∂D:

$$\frac{\partial u}{\partial \nu(x)} + \frac{1}{2\pi}\int_{\partial D} \frac{\partial}{\partial \nu(x)} G(\underset{\sim}{x},\underset{\sim}{y}) \frac{\partial u}{\partial \nu(y)} ds(y)$$

$$= 2 \frac{\partial u^i}{\partial \nu(x)} - \frac{\partial}{\partial \nu(x)} \frac{1}{2\pi i}\int_{\partial B} u^i \frac{\partial}{\partial \nu(y)} G(\underset{\sim}{x},\underset{\sim}{y}) \, ds(y) \; ; \; \underset{\sim}{x} \in \partial D \;. \tag{2.6}$$

Note that in the special case where $u^i(\underset{\sim}{x}) = 1$ we have

$$\frac{1}{2\pi} \int_{\partial B} \frac{\partial}{\partial \nu(y)} G(\underset{\sim}{x},\underset{\sim}{y})\, ds(y) = 2\, \frac{R}{|\underset{\sim}{x}|} \tag{2.7}$$

for $\underset{\sim}{x} \in \partial D$ and hence in this case the right hand side of (2.6) becomes $-2\, R\, \frac{\partial}{\partial \nu} \frac{1}{|\underset{\sim}{x}|}$.

We shall now show that the integral equation (2.6) can be solved by successive approximations. (The solution of (2.1) can then be obtained by substituting the solution of (2.6) into (2.5), i.e. we have a constructive method for solving (2.1).) To this end let $C(\partial D)$ denote the Banach space of continuous complex valued functions defined on ∂D and equipped with the maximum norm and define the compact integral operator $\underset{\sim}{K} : C(\partial D) \to C(\partial D)$ by

$$(\underset{\sim}{K}v)(\underset{\sim}{x}) = -\frac{1}{2\pi} \int_{\partial D} \frac{\partial}{\partial \nu(x)} G(\underset{\sim}{x},\underset{\sim}{y})\, v(\underset{\sim}{y})\, ds(y) . \tag{2.8}$$

If we let $f(\underset{\sim}{x})$ be the right hand side of (2.6) then (2.6) can be written as

$$v - \underset{\sim}{K}v = f \tag{2.9}$$

where $v = \partial u/\partial \nu$. In order to show that (2.9) can be solved by iteration we shall show that the spectrum of $\underset{\sim}{K}$ is contained in $(-1,1)$. We first note that since $\underset{\sim}{K}$ is compact all spectral values of $\underset{\sim}{K}$ different from zero are eigenvalues. Let λ be an eigenvalue of $\underset{\sim}{K}$ corresponding to the eigenfunction ϕ, $K\phi = \lambda\phi$, and define

$$u(\underset{\sim}{x}) = \frac{1}{2\pi} \int_{\partial D} G(\underset{\sim}{x},\underset{\sim}{y})\,\phi(\underset{\sim}{y})\; ds(y) \;;\; \underset{\sim}{x} \in \mathbb{R}^3 \backslash B \;. \tag{2.10}$$

Then u is harmonic in $\mathbb{R}^3\backslash\bar{D}$ and $D\backslash\bar{B}$, and from the jump discontinuity properties of single layer potentials we have

$$u_+ = u_- \quad \text{on} \quad \partial D \tag{2.11a}$$

$$\begin{aligned} \frac{\partial u_\pm}{\partial \nu} &= - \underset{\sim}{K}\,\phi \mp \phi \\ &= (-\lambda \mp 1)\phi \qquad \text{on} \quad \partial D \end{aligned} \tag{2.11b}$$

where $\pm$ denote the limits from outside and inside D respectively. Hence

$$(1-\lambda)\,\frac{\partial u_+}{\partial \nu} + (1+\lambda)\,\frac{\partial u_-}{\partial \nu} = 0 \tag{2.12}$$

and since $u(\underset{\sim}{x}) = 0$ for $\underset{\sim}{x} \in \partial B$ we have from (2.11a), (2.12) and Green's theorem that

$$\begin{aligned} 0 &= (1-\lambda) \int_{\partial D} \bar{u}_+ \frac{\partial u_+}{\partial \nu}\, ds + (1+\lambda) \int_{\partial D} \bar{u}_- \frac{\partial u_-}{\partial \nu}\, ds - (1+\lambda) \int_{\partial B} \bar{u}\, \frac{\partial u}{\partial \nu}\, ds \\ &= (\lambda-1) \iint_{\mathbb{R}^3\backslash D} |\text{grad } u|^2 dx + (\lambda+1) \int_{D\backslash B} |\text{grad } u|^2 dx \;. \end{aligned} \tag{2.13}$$

Now define

$$\begin{aligned} I(u) &= \iint_{D/B} |\text{grad } u|^2 dx \\ \hat{I}(u) &= \iint_{\mathbb{R}^3\backslash D} |\text{grad } u|^2 dx \;. \end{aligned} \tag{2.14}$$

Then if $\hat{I}(u) + I(u) > 0$ we have

$$\lambda = \frac{\hat{I}(u) - I(u)}{\hat{I}(u) + I(u)} \tag{2.15}$$

which implies that in this case all the eigenvalues of $\underset{\sim}{K}$ are real and contained in $(-1,1)$. If $\hat{I}(u) + I(u) = 0$ then u is constant in $\mathbb{R}^3\backslash\bar{D}$ and $D\backslash B$ and since from (2.10) we have that u vanishes at infinity and on ∂B we can conclude that $u(\underset{\sim}{x}) = 0$ for $x \in \mathbb{R}^3\backslash B$. Hence, from (2.11b) we have

$$\phi = \frac{1}{2}\left(\frac{\partial u_-}{\partial \nu} - \frac{\partial u_+}{\partial \nu}\right) = 0 \ , \tag{2.16}$$

i.e. λ is not an eigenvalue, contrary to assumption. Hence, the spectrum of $\underset{\sim}{K}$ is contained in $(-1,1)$ and hence (2.9) can be solved by successive approximations.

Before turning our attention to the exterior Dirichlet problem for the Helmholtz equation we briefly consider what effect a change of the radius R of the ball B has on the spectral radius of the operator $\underset{\sim}{K}$. In particular, consider the special case when D is a ball of radius one and u is defined by (2.10). Then u has an expansion in spherical harmonics of the form

$$u(\underset{\sim}{x}) = \begin{cases} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (a_{nm}\, \rho^{n} + b_{nm}\, \rho^{-n-1})\, P_n^m (\cos\theta) e^{im\phi}; & R \le \rho \le 1 \\ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_{nm}\, \rho^{-n-1}\, P_n^m (\cos\theta) e^{im\phi}; & 1 \le \rho \le \infty \end{cases} \tag{2.17}$$

where (ρ,θ,ϕ) are spherical coordinates, P_n^m denotes an associated Legendre polynomial, and the a_{nm}, b_{nm}, c_{nm} are constants. Then (2.11a), (2.12) and the fact that $u(\underset{\sim}{x}) = 0$ for

$x \in \partial B$ implies that

$$a_{nm} + b_{nm} - c_{nm} = 0$$

$$R^{2n+1} a_{nm} + b_{nm} = 0 \qquad (2.18)$$

$$(1+\lambda)n\, a_{nm} - (1+\lambda)(n+1)\, b_{nm} - (1-\lambda)(n+1) c_{nm} = 0$$

and this system has a nontrivial solution if and only if the determinant of the coefficients vanishes, i.e.

$$\lambda = \lambda_n = \frac{1-(2n+2)R^{2n+1}}{2n+1} \quad . \qquad (2.19)$$

From (2.19) we see that in this special case the eigenvalues decrease as R increases, i.e. by using overrelaxation we can make the spectral radius smaller. This in turn implies that the rate of convergence of the successive approximations to the solution of the integral equation is improved. By using the methods introduced in [13] it can in fact be shown that this behaviour is true not only for a sphere but also for an arbitrary bounded simply connected domain ([11], [19]). Hence for computational purpose it is desirable to pick R as large as possible and then use overrelaxation to improve the rate of convergence (c.f. [13]).

We now consider the exterior Dirichlet problem for the Helmholtz equation, i.e. to find $u \in C^2(\mathbb{R}^3\backslash\bar{D}) \cap C^o(\mathbb{R}^3\backslash D)$ such that

$$u(\underset{\sim}{x}) = u^i(\underset{\sim}{x}) + u^s(\underset{\sim}{x}) \quad \text{in} \quad \mathbb{R}^3 \backslash D \tag{2.20a}$$

$$\Delta_3 u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^3 \backslash \bar{D} \tag{2.20b}$$

$$u(\underset{\sim}{x}) = 0 \quad \text{for} \quad \underset{\sim}{x} \in \partial D \tag{2.20c}$$

$$\lim_{r\to\infty} r \left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{2.20d}$$

where the "incoming wave" u^i is a solution of (2.20b) in all of $\mathbb{R}^3$, the wave number k is positive, and the radiation condition (2.20d) for the "scattered wave" u^s is assumed to hold uniformly in all directions. The domain D is assumed to satisfy the same conditions as in problem (2.1) and under these conditions we can again conclude that a solution to (2.20) exists and $u \in C^2(\mathbb{R}^3\backslash\bar{D}) \cap C^1(\mathbb{R}^3\backslash D)$ ([18]). Now define

$$G_k(\underset{\sim}{x},\underset{\sim}{y}) = \frac{e^{ik|\underset{\sim}{x}-\underset{\sim}{y}|}}{|\underset{\sim}{x}-\underset{\sim}{y}|} - \frac{R}{|\underset{\sim}{x}|}\,\frac{e^{ik|\underset{\sim}{x}^*-\underset{\sim}{y}|}}{|\underset{\sim}{x}^*-\underset{\sim}{y}|} \; ; \; \underset{\sim}{x}^* = \frac{R^2}{|\underset{\sim}{x}|^2}\,\underset{\sim}{x} \tag{2.21}$$

where R is again the radius of the ball B contained in the interior of D. Then with respect to $\underset{\sim}{y}$ the function G_k is a fundamental solution to the Helmholtz equation satisfying the radiation condition (2.20d). Hence, proceeding in the same manner as we did for Laplace's equation, we can use Green's formula to derive the following integral equation for the unknown normal derivative of the solution to (2.20) evaluated on ∂D:

$$\frac{\partial u}{\partial\nu(x)} + \frac{1}{2\pi}\int_{\partial D} \frac{\partial}{\partial\nu(x)} G_k(\underset{\sim}{x},\underset{\sim}{y}) \frac{\partial}{\partial\nu(y)} ds(y) = 2\,\frac{\partial u^i}{\partial\nu(x)}$$

$$+ \frac{\partial}{\partial\nu(x)}\,\frac{1}{2\pi}\int_{\partial B} \left\{\frac{\partial u^i}{\partial\nu(y)} G_k(\underset{\sim}{x},\underset{\sim}{y}) - u^i \frac{\partial}{\partial\nu(y)} G_k(\underset{\sim}{x},\underset{\sim}{y})\right\} ds(y); \; \underset{\sim}{x} \in \partial D. \tag{2.22}$$

Note that the additional term on the right hand side of (2.22) (as compared with the right hand side of (2.26)) occurs since we no longer have the fact that $G_k(\underset{\sim}{x},\underset{\sim}{y}) = 0$ for $y \in \partial B$. This minor inconvenience could have been avoided by chosing G_k to be the Green's function for the Helmholtz equation in the exterior of B instead of as in (2.21), but in this case the kernel of the integral equation (2.22) could no longer be expressed in a simple closed form such as (2.21).

We shall now show that the integral equation (2.22) can be solved by iteration for k sufficiently small. As in the case of Laplace's equation this provides a constructive method for solving the boundary value problem (2.20). Let $C(\partial D)$ again denote the Banach space of continuous complex valued functions defined on ∂D equipped with the maximum norm and define the compact integral operator $\underset{\sim}{K}_k : C(\partial D) \to C(\partial D)$ by

$$(\underset{\sim}{K}_k v)(\underset{\sim}{x}) = -\frac{1}{2\pi}\int_{\partial D} \frac{\partial}{\partial\nu(x)} G_k(\underset{\sim}{x},\underset{\sim}{y})\, v(\underset{\sim}{y})\, ds(y) \quad . \qquad (2.23)$$

Let $f(\underset{\sim}{x})$ denote the right hand side of (2.22). Then the integral equation (2.22) can be written in the form

$$v - \underset{\sim}{K}_k v = f \qquad (2.24)$$

where $v = \frac{\partial u}{\partial \nu}$. Then since

$$|G_k(\underset{\sim}{x},\underset{\sim}{y}) - G(\underset{\sim}{x},\underset{\sim}{y})| = O(k) \qquad (2.25)$$

it follows that

$$||\underset{\sim}{K}_k - \underset{\sim}{K}|| = O(k) \tag{2.26}$$

where $||\cdot||$ denotes the maximum operator norm. Hence, since the spectral radius of $\underset{\sim}{K}$ is less than one, we can follow the argument of Theorem 2.2 of [10] to conclude that there exists a positive constant k_0 such that the spectral radius of $\underset{\sim}{K}_k$ is less than one for $0 \leq k \leq k_0$. Thus for such values of k we can solve (2.22) by the method of successive approximations.

III. Remarks on the Inverse Scattering Problem

We now consider the application of the results of Section Two to the inverse scattering problem associated with (2.20). In particular, if we apply Green's formula to the solution u of (2.20) and the fundamental solution for the Helmholtz equation, we have that for $\underset{\sim}{x} \in \mathbb{R}^3 \backslash \overline{D}$ (c.f. [6])

$$\begin{aligned} u^s(\underset{\sim}{x}) &= -\frac{1}{4\pi}\int_{\partial D} \frac{\partial u}{\partial \nu(y)} \frac{e^{ik|\underset{\sim}{x}-\underset{\sim}{y}|}}{|\underset{\sim}{x}-\underset{\sim}{y}|}\, ds(y) \\ &= \frac{e^{ikr}}{4\pi r} F(\theta,\phi;k) + O\left(\frac{1}{r^2}\right) \end{aligned} \tag{3.1}$$

where (r,θ,ϕ) are spherical coordinates and

$$F(\theta,\phi;k) = -\frac{1}{4\pi}\int_{\partial D} \frac{\partial u}{\partial \nu(y)} e^{-ik\underset{\sim}{\eta}\cdot\underset{\sim}{y}}\, ds(y); \quad \underset{\sim}{\eta} = \frac{\underset{\sim}{x}}{|\underset{\sim}{x}|}. \tag{3.2}$$

The *inverse scattering problem* we wish to consider is to determine the shape of ∂D from a knowledge of the far field pattern F for all angles θ,ϕ and low values of the wave number k.

We first want to reduce the solution of the inverse scattering problem to a generalized moment problem for the conductor potential of the domain D. To this end suppose $u^i(\underset{\sim}{x}) = e^{ikx_1}$ where $\underset{\sim}{x} = (x_1,x_2,x_3)$. Then from the results of Section Two, we have that the integral equation (2.24) can be solved by successive approximations, i.e.

$$\frac{\partial u}{\partial \nu(x)} = \sum_{n=0}^{\infty} \underset{\sim}{K}_k^n \left[-2R \frac{\partial}{\partial \nu} \frac{1}{|\underset{\sim}{x}|}\right] + O(k) \quad . \tag{3.2}$$

From (2.26) and [17], p. 164, we now have from (3.2) that

$$\begin{aligned} \frac{\partial u}{\partial \nu(x)} &= \sum_{n=0}^{\infty} \underset{\sim}{K}_0^n \left[-2R \frac{\partial}{\partial \nu} \frac{1}{|\underset{\sim}{x}|}\right] + O(k) \\ &= \frac{\partial u_0}{\partial \nu(x)} + O(k) \end{aligned} \tag{3.3}$$

for $\underset{\sim}{x} \in \partial D$. From the results of Section Two we can now identify $u_0(\underset{\sim}{x})$ for $x \in \mathbb{R}^3 \backslash D$ as the solution of the Dirichlet problem

$$u_0(\underset{\sim}{x}) = 1 - u_0^s(\underset{\sim}{x}) \quad \text{in} \quad \mathbb{R}^3 \backslash D \tag{3.4a}$$

$$\Delta_2 u_0 = 0 \quad \text{in} \quad \mathbb{R}^3 \backslash \bar{D} \tag{3.4b}$$

$$u_0(\underset{\sim}{x}) = 0 \quad \text{for} \quad \underset{\sim}{x} \in \partial D \tag{3.4c}$$

$$\lim_{|x| \to \infty} u_0^s(\underset{\sim}{x}) = 0 \ , \tag{3.4d}$$

i.e. u_0^s is the conductor potential of the surface ∂D (c.f. [3]). Now expand F in a series of spherical harmonics

$$F(\theta,\phi;k) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm}(k) P_n^m(\cos \theta)\, e^{im\phi} \tag{3.5}$$

and write $\underset{\sim}{\eta}$ and $\underset{\sim}{y}$ in spherical coordinates as

$$\begin{aligned} \underset{\sim}{\eta} &= (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) \\ \underset{\sim}{y} &= \rho(\sin\theta'\cos\phi',\ \sin\theta'\sin\phi',\ \cos\theta'). \end{aligned} \tag{3.6}$$

Then

$$\begin{aligned} \underset{\sim}{\eta}\cdot\underset{\sim}{y} &= \rho[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')] \\ &= \rho\cos\gamma \end{aligned} \tag{3.7}$$

and since ([14])

$$e^{-ik\rho\cos\gamma} = \sqrt{\frac{\pi}{2k\rho}} \sum_{n=0}^{\infty} (-i)^n (2n+1) J_{n+\frac{1}{2}}(k\rho)\, P_n(\cos\gamma) \tag{3.8a}$$

$$P_n(\cos\gamma) = \sum_{m=-n}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)\, P_n^m(\cos\theta')\, e^{im(\phi-\phi')} \tag{3.8b}$$

where $J_{n+\frac{1}{2}}$ denotes Bessel's function and P_n Legendre's polynomial we have from (3.2), (3.5), (3.8) and the orthogonality of the associated Legendre polynomials that

$$\begin{aligned} a_{nm}(k) &= \int_0^{2\pi}\int_0^{\pi} F(\theta,\phi;k)\, P_n^m(\cos\theta)\sin\theta\, e^{-im\phi}\, d\theta d\phi \\ &= -\frac{1}{4\pi}\int_{\partial D} \frac{\partial u}{\partial\nu(y)} \left[\int_0^{2\pi}\int_0^{\pi} e^{-ik\rho\cos\gamma} P_n^m(\cos\theta)\sin\theta\, e^{-im\phi}\, d\theta d\phi\right] ds(y) \\ &= -\sqrt{\frac{\pi}{2}}\, i^{-n} \int_{\partial D} \frac{\partial u}{\partial\nu(y)} \left[\frac{1}{\sqrt{k\rho}} J_{n+\frac{1}{2}}(k\rho)\, P_n^m(\cos\theta')e^{-im\phi'}\right] ds(y) \quad . \end{aligned} \tag{3.9}$$

Since

$$\frac{1}{\sqrt{k\rho}} J_{n+1/2}(k\rho) = \frac{(k\rho)^n}{2^{n+1/2}\,\Gamma(n+3/2)} + O(k\rho)^{n+2} \tag{3.10}$$

we have from (3.3), (3.9) and (3.10) that for $n \geq 0$, $-n \leq m \leq n$,

$$a_{nm}(k) = -\frac{i^{-n}}{2^{n+1}}\frac{\sqrt{\pi}\,k^n}{\Gamma(n+3/2)}\int_{\partial D}\frac{\partial u_0}{\partial\nu(y)}\left[\rho^n P_n^m(\cos\theta')e^{-im\phi'}\right]ds(y) + O(k^{n+1}). \tag{3.11}$$

Hence, if we define

$$\mu_{nm} = \lim_{k\to 0}\left[\frac{2^{n+1}\,\Gamma(n+3/2)\,i^n\,a_{nm}(k)}{\sqrt{\pi}\,k^n}\right] \tag{3.12}$$

then the inverse scattering problem has been reduced to determining D from the "moment" problem

$$\mu_{nm} = -\int_{\partial D}\frac{\partial u_0}{\partial\nu(y)}\left[\rho^n\,P_n^m\,(\cos\theta')\,e^{-im\phi'}\right]ds(y)\ . \tag{3.13}$$

Note that although the μ_{nm} are explicitly computable from the far field pattern F, small errors in measuring the coefficients $a_{nm}(k)$ will result in large errors in the numbers μ_{nm} if n is large. Hence, in practice we must assume that only a finite number of the μ_{nm} are known. Observe also that the "moment" problem (3.13) is nonlinear since both u_0 and the region of integration depend on D.

We shall now show that (3.13) allows us to compute u_0 outside of a ball $\hat{B}$ containing D in its interior. Since only a finite number of the μ_{nm} are known we can only compute u_0 approximately, and in order to obtain error estimates it is necessary to know the radius of $\hat{B}$ "a priori". Having obtained an

approximation to u_0, we note that the level curve $u_0(\underset{\sim}{x}) = 0$ is ∂D and hence the level curves $u_0(\underset{\sim}{x}) = \gamma$, $0 < \gamma < 1$, approximate ∂D as γ tends to zero. Thus, if we determine the curves $u_0(\underset{\sim}{x}) = \gamma$, $0 < \gamma < 1$, for $\underset{\sim}{x} \in \mathbb{R}^3 \backslash \hat{B}$ and extrapolate as γ tends to zero we can determine an approximation to ∂D. Obviously the success of this procedure is optimized if $\hat{B}$ is in fact the smallest ball containing D in its interior. To determine an approximation to $u_0(\underset{\sim}{x})$ for $\underset{\sim}{x} \in \mathbb{R}^3 \backslash \hat{B}$ we use the fact that $u_0(\underset{\sim}{x}) = 0$ for $\underset{\sim}{x} \in \partial D$ and hence from (3.13) and Green's formula we have

$$\mu_{nm} = \int_{\partial \hat{B}} \left\{ u_0 \frac{\partial}{\partial \rho} [\rho^n P_n^m (\cos \theta') e^{-im\phi'}] - \frac{\partial u_0}{\partial \rho} [\rho^n P_n^m (\cos \theta') e^{-im\phi'}] \right\} ds. \tag{3.14}$$

Since on $\partial \hat{B}$, u_0 has an expansion of the form

$$u_0(\underset{\sim}{x}) = 1 + \sum_{n=0}^{\infty} \sum_{m=-m}^{n} b_{nm} \rho^{-n-1} P_n^m (\cos \theta) e^{im\phi} \tag{3.15}$$

we have from (3.14), (3.15) and the orthogonality of the associated Legendre polynomials that

$$\mu_{nm} = 4\pi b_{nm} \frac{(n+m)!}{(n-m)!} \tag{3.16}$$

Suppose now that we know the μ_{nm} (and hence from (3.16) the b_{nm}) for $-n \leq m \leq n$, $0 \leq n \leq N$. We then want to know how accurately

$$u_0^N(\underset{\sim}{x}) = 1 + \sum_{n=0}^{N} \sum_{m=-n}^{n} b_{nm} \rho^{-n-1} P_n^m (\cos \theta) e^{im\phi} \tag{3.17}$$

approximates $u_0(\underset{\sim}{x})$ for $\underset{\sim}{x} \in \mathbb{R}^3 \setminus \hat{B}$. To this end we note that from (3.15) we have that for $n \geq 1$ and $\rho \geq a$ = radius of $\hat{B}$,

$$\frac{4\pi}{2n+1} \frac{(n+m)!}{(n-m)!} b_{nm} \rho^{-n-1} =$$

$$\int_0^{2\pi}\int_0^{\pi} u_0(\underset{\sim}{x}) P_n^m (\cos\theta) e^{-im\phi} \sin\theta \, d\theta d\phi \tag{3.18}$$

and since from the maximum principle $0 \leq u_0(\underset{\sim}{x}) \leq 1$ for $\underset{\sim}{x} \in \mathbb{R}^3 \setminus D$, we have from (3.18) that

$$\left| \frac{4\pi (n+m)!}{(2n+1)(n-m)!} b_{nm} a^{-n-1} \right| \leq 2\pi \int_0^{\pi} |P_n^m(\cos\theta)| \sin\theta \, d\theta$$

$$\leq 2\pi \left[\int_0^{\pi} \sin\theta \, d\theta\right]^{1/2} \left[\int_0^{\pi} |P_n^m(\cos\theta)|^2 \sin\theta \, d\theta\right]^{1/2}$$

$$= 2^{3/2}\pi \sqrt{\frac{4\pi (n+m)!}{(2n+1)(n-m)!}} , \tag{3.19}$$

i.e.

$$|b_{nm}| \leq a^{n+1} \sqrt{\frac{2\pi(2n+1)(n-m)!}{(n+m)!}} . \tag{3.20}$$

Hence, for $\rho > a$ we have

$$\frac{1}{4\pi\rho^2} \int_{|\underset{\sim}{x}|=\rho} |u_0(\underset{\sim}{x}) - u_0^N(\underset{\sim}{x})|^2 ds = \sum_{n=N+1}^{\infty} \sum_{m=-n}^{n} |b_{nm}|^2 \frac{(n+m)!\rho^{-2n-2}}{(n-m)!(2n+1)} \tag{3.21}$$

$$\leq 2\pi \sum_{n=N+1}^{\infty} (2n+1) \left(\frac{a}{\rho}\right)^{2n+2}$$

and in view of (3.16) this tells us how many Fourier coefficients of the far field pattern are needed in order to approximate

$u_0(\underset{\sim}{x})$ for $|\underset{\sim}{x}| \geq a_0 > a$ to obtain a given degree of accuracy in the L^2 sense. We note that considerably sharper results are obtainable in the case of the inverse scattering problem for a cylinder due to the fact that conformal mapping techniques are now available (c.f. [4], [5]).

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The low frequency inverse scattering problem in R^3 is studied subject to Dirichlet boundary conditions and acoustic wave propagation.